On the approximation of $\Xi_\tau(z)$ in the ULP model

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In our paper on "Random Tranches" (*Risk*, March 2003), we define the function

$$\Xi_\tau(z) \equiv \int_0^1 B(z; \tau u, \tau(1-u))du \tag{1}$$

where $B(z; a, b)$ denotes the Beta(a, b) cumulative distribution function evaluated at z. This function does not have an analytical solution. In order to arrive at a tractable and robust solution to our "Uncertainly in Loss Prioritization" (ULP) model, we require a reasonably simple and highly accurate approximation to $\Xi_\tau(z)$. This technical note develops such an approximation and tests its accuracy over the feasible range of values for τ.

Analytic solutions to $\Xi_\tau(z)$ are available in the special cases of $\tau = 0$ and $\tau = \infty$. For all $z \in (0, 1)$ and $u \in (0, 1)$, we have

$$\lim_{\tau\to 0} B(z; \tau u, \tau(1-u)) = 1 - u,$$

which implies $\Xi_0(z) = 1/2$. At the other extreme, we have

$$\lim_{\tau\to\infty} B(z; \tau u, \tau(1-u)) = \mathbb{1}_{\{z \geq u\}},$$

which implies $\Xi_\infty(z) = z$. It is desirable that our approximation to Ξ_τ take on the same limiting forms.

For large but finite values of τ, the $\Xi_\tau(z)$ function is close to z. The difference cannot be ignored, however, as it represents the impact of uncertainty in loss prioritization. Figure 1 shows how $\Xi_\tau(z)$ varies with τ. By substracting z from $\Xi_\tau(z)$, we can better focus on the component of interest. For positive finite τ, $\Xi_\tau(z)$ weaves around the $45°$ line in a regular symmetric pattern. As shown in Figure 2, the function $\Xi_\tau(z) - z$ starts at zero, rises sharply, levels off quickly, then becomes roughly linear with negative slope and hits zero at $z = 1/2$. The pattern above $z = 1/2$ is the mirror image of the pattern below $z = 1/2$; i.e., the function displays rotational symmetry around $z = 1/2$.

A simple function that displays the same cyclical behavior is $(1/2 - z)(z(1-z))^{(\alpha-1)}$ for $\alpha \geq 1$.

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Therefore, we propose to approximate $\Xi_\tau(z)$ by

$$\hat{\Xi}_\tau(z) \equiv z + \xi\left(\frac{1}{2} - z\right)\frac{(z(1-z))^{\alpha-1}}{B(\alpha,\alpha)}, \tag{2}$$

where the coefficients ξ and α are functions of τ. Weighting $(z(1-z))^{\alpha-1}$ by the beta function is natural as it transforms the last piece of $\hat{\Xi}_\tau$ into a beta density.

We solve for coefficients ξ and α through moment matching. The Ξ_τ function can be understood as the unconditional cdf of a random variable Z that has conditional distribution $Z|U \sim \mathrm{Beta}(\tau U, \tau(1-U))$, where $U \sim \mathrm{Uniform}[0,1]$. The approximation $\hat{\Xi}_\tau(z)$ equals zero at $z=0$, one at $z=1$, and is increasing in between, so also can be treated as a cdf on the unit interval. In each case, the first moment is $1/2$. We set ξ and α so that the second and fourth moments of the two distributions match. (Due to the rotational symmetry of Ξ and $\hat{\Xi}$, the third moments add no new information.)

As Ξ_τ is a compounded beta distribution, its moments are easily obtained. The j^{th} uncentered moment is given by

$$\lambda_j \equiv \frac{1}{(\tau)_j}\int_0^1 (\tau u)_j du$$

where $(a)_k$ is Pochhammer's notation, i.e., $(a)_0 = 1$, $(a)_1 = a$, $(a)_k = (a)_{k-1}(a+k-1)$. The function $(\tau u)_j$ is merely a j^{th} order polynomial in u, so λ_j has a simple closed form solution for any j. The corresponding moments for $\hat{\Xi}_\tau$ also have closed-form solution:

$$\hat{\lambda}_j \equiv \frac{1}{j+1} + \xi\frac{j(j-1)}{2}\frac{(\alpha)_{j-1}}{(2\alpha)_j}.$$

We set $\lambda_j = \hat{\lambda}_j$ for $j=2$ and $j=4$, and solve for ξ and α:

$$\alpha = \frac{3(\tau^2+6\tau+6)}{3\tau^2+13\tau+18} \qquad \text{and} \qquad \xi = \frac{2\alpha+1}{3(\tau+1)}.$$

The approximation is extraordinarily precise over the entire range of τ values. In the four panels of Figure 3, we plot $\Xi_\tau(z) - z$ and $\hat{\Xi}_\tau(z) - z$ for $\tau = (1, 8, 64, 512)$. Subtracting out the linear component serves to heighten the visual differences between Ξ and our approximation, yet in each case the fit is nearly perfect. The approximation also satisfies the desired limiting behavior. When $\tau = 0$, $\alpha = \xi = 1$, so $\hat{\Xi}_\tau(z) = 1/2$. When $\tau = \infty$, $\xi = 0$ and $\alpha = 1$, so $\hat{\Xi}_\tau(z) = z$.

For reasonably large values of τ, we can approximate $\alpha \approx 1$ and $\xi \approx 1/\tau$. Figure 4 shows how α and ξ vary with τ. In the upper panel, we see that $\alpha(\tau)$ is nonlinear for low values of τ, but asymptotes to one as τ heads towards infinity (note the log-scale on the τ axis). In the lower panel, we see that $\xi(\tau)$ converges quite closely to $1/\tau$ by $\tau = 100$ (log-scale on both axes). When we apply these approximations, we arrive at the simple functional form

$$\hat{\Xi}_\tau(z) = z + \frac{1}{\tau}\left(\frac{1}{2} - z\right).$$

Figure 1: Dependence of $\Xi_\tau(z)$ on τ

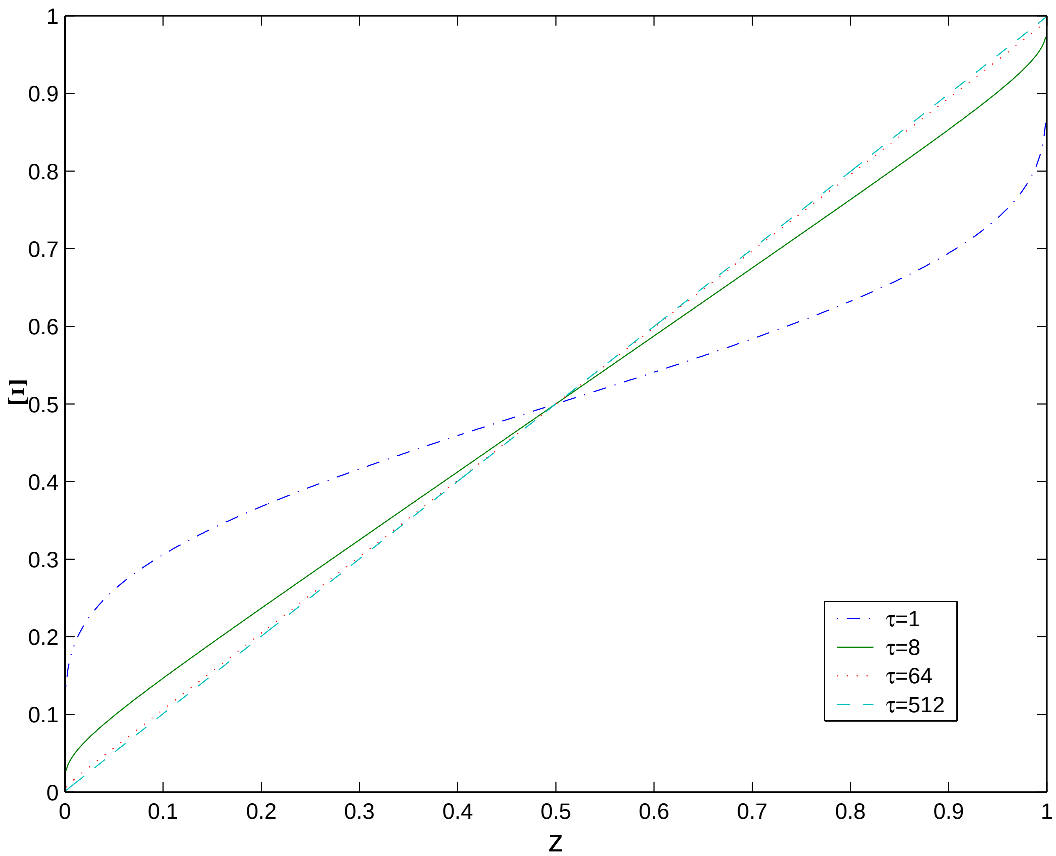


Figure 2: Cyclical component of $\Xi_{\tau}(z)$

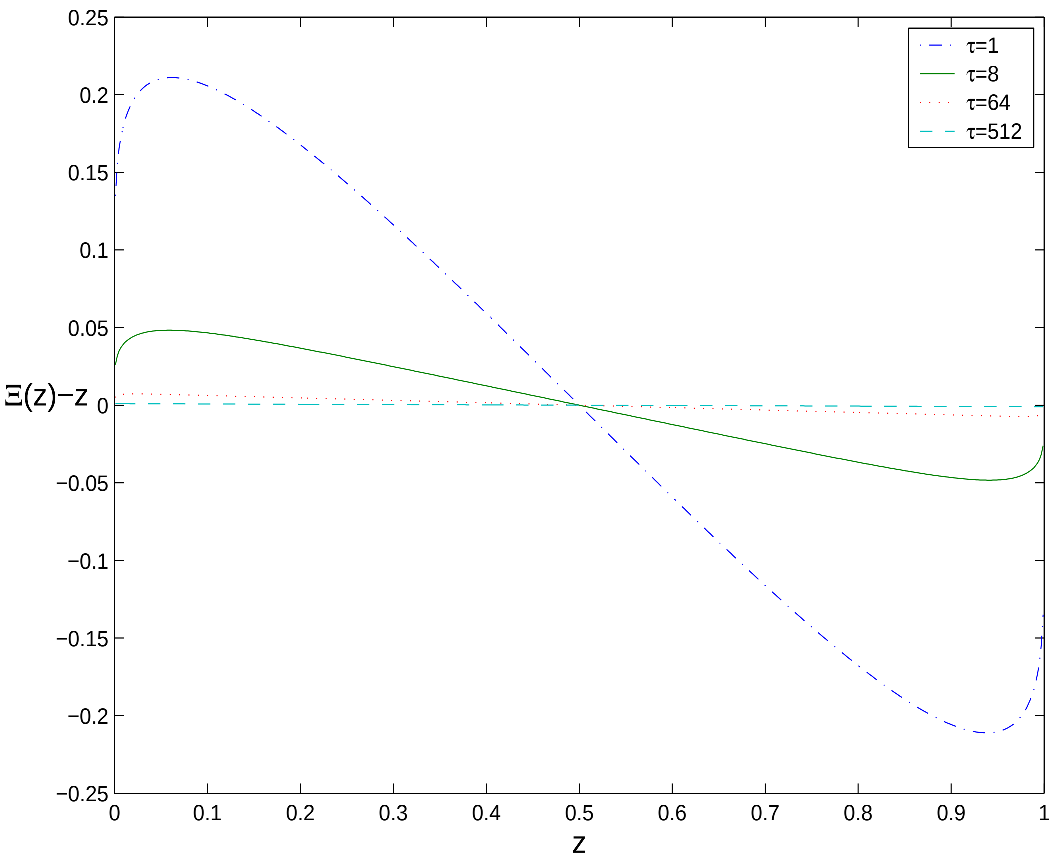
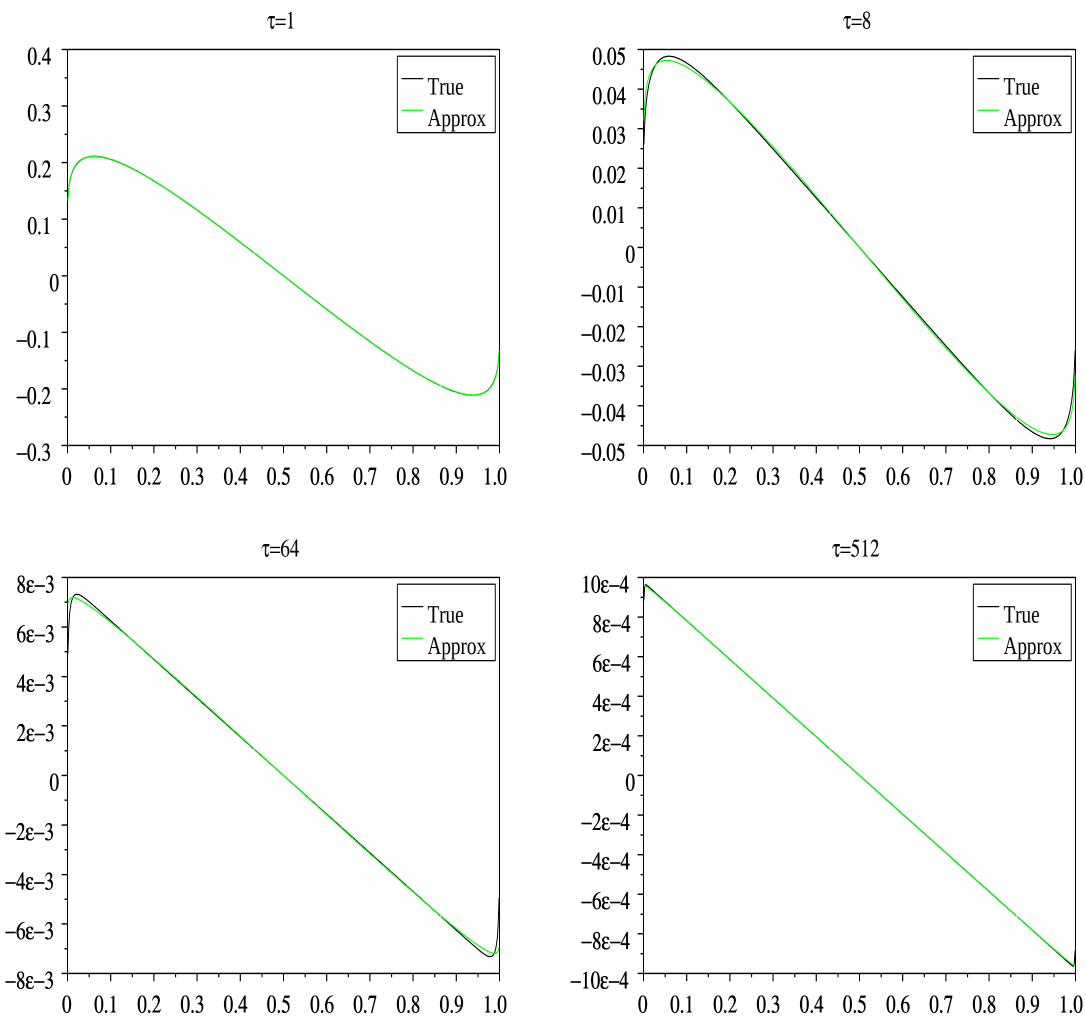


Figure 3: Cyclical components of $\Xi_\tau(z)$ and $\hat{\Xi}_\tau(z)$



Note: The panels show $\Xi_\tau(z) - z$ and $\hat{\Xi}_\tau(z) - z$ for different values of τ.

Figure 4: Coefficients α and ξ as functions of τ